A Regime-Switching Structural Equation Modeling Framework for Formulating Multi-Phase Linear and Nonlinear Growth Curve Models

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MODS LAB MODELING DEVELOPMENTAL SYSTEMS

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Outline of My Talk

1. Single-phase, linear and nonlinear growth curve models
2. Multiphase, sequential growth curve models
3. Regime-switching linear and nonlinear growth curve models

Single-Phase Growth Curve Models

Linear Growth Curve Model

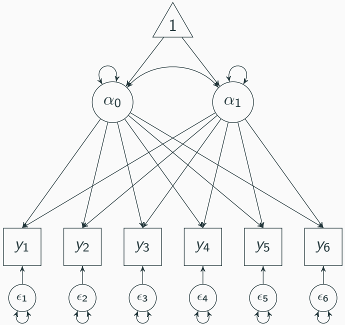

Figure 1: Linear growth curve model

Loadings shown as solid arrows = loadings fixed at known constant values

Between-individual differences (IDs) in intercept and linear slope are the key sources of IDs

$$
\begin{aligned}
y_{it} &= \alpha_{0i} + \alpha_{1i} t_{ij} + \epsilon_{ij}, \quad (1)\\
\alpha_{0i} &= \mu_{\alpha_0} + u_{\alpha_0,i}\\
\alpha_{1i} &= \mu_{\alpha_1} + u_{\alpha_1,i}
\end{aligned}
$$

$$
\begin{bmatrix} u_{\alpha_0,i} \\ u_{\alpha_1,i} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2_{\alpha_0} & \\ \sigma_{\alpha_{01}} & \sigma^2_{\alpha_1} \end{bmatrix} \right)
$$

Latent Basis Growth Curve Model (McArdle & Epstein, 1987; Meredith & Tisak, 1984, 1990)

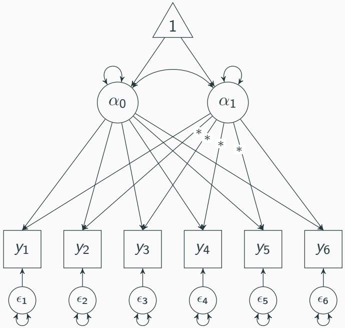

Figure 2: Latent basis growth curve model
$*$ = freely estimated parameters

- Group-based change point may be reflected implicitly through b_j
- α_{1i} captures some IDs in deviations from the group-based change trajectory.

$$\begin{aligned} y_{ij} &= \alpha_{0i} + \alpha_{1i} f(t_j) + \epsilon_{ij}, \qquad (2) \\ f_j(t_j) &= \begin{cases} t_j, \text{for at least} \\ \text{two time points} \\ b_j, \text{for all remaining } j \end{cases} \\ b_j &= \text{freely estimated parameters.} \end{aligned}$$

Latent Class Growth Curve or Growth Mixture Models

(Muthén, 2001; Nagin, 1999)

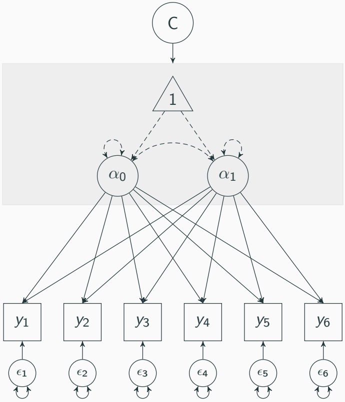

Figure 3: Growth mixture model
Loadings shown as solid arrows = loadings fixed at known constant values

- Dashed arrows in shaded box mark parameters that are typically allowed to vary by class.

$$\begin{aligned} y_{ij} &= \alpha_{\mathbf{0},C_i} + \alpha_{\mathbf{1},C_i} t_{ij} + \epsilon_{ij}, \\ \alpha_{\mathbf{0}C_i} &= \mu_{\alpha_{\mathbf{0}},C_i} + u_{\alpha_{\mathbf{0}},C_i} \\ \alpha_{\mathbf{1}C_i} &= \mu_{\alpha_{\mathbf{1}},C_i} + u_{\alpha_{\mathbf{1}},C_i} \\ \begin{bmatrix} u_{\alpha_{\mathbf{0}},C_i} \\ u_{\alpha_{\mathbf{1}},C_i} \end{bmatrix} &\sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2_{\alpha_{\mathbf{0}},C_i} & \\ \sigma_{\alpha_{\mathbf{01}},C_i} & \sigma^2_{\alpha_{\mathbf{1}},C_i} \end{bmatrix} \right) \end{aligned} \tag{3}$$

C_i = a latent class indicator that indicates individual i's membership in L possible classes.

Structured Latent Curve Model

(Browne, 1993; Browne & du Toit, 1991)

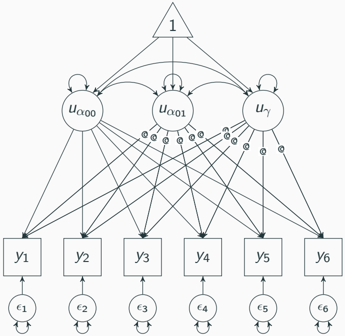

Figure 4: Exponential growth curve model
@ = loadings composed of constraint functions with estimated parameters

- An approach for fitting growth curve functions that may be nonlinear in some unit- (e.g., person-) specific parameters, $\boldsymbol{\lambda}_i$.

$$\begin{aligned} y_{ij} &= f(\boldsymbol{\lambda}_i, t_{ij}) + \epsilon_{ij}, \\ \boldsymbol{\lambda}_i &= \boldsymbol{\lambda}_0 + \boldsymbol{u}_i \end{aligned} \tag{4}$$

First-order Taylor series expansion of Equation (4) around $\boldsymbol{\lambda}_0$ yields:

$$\begin{aligned} f(\boldsymbol{\lambda}_i, t_{ij}) &\approx f(\boldsymbol{\lambda}_0, t_{ij}) \\ &+ \frac{d}{d\boldsymbol{\lambda}_0} f(\boldsymbol{\lambda}_0, t_{ij})(\boldsymbol{\lambda}_i - \boldsymbol{\lambda}_0) \\ &= \mu_{y_{ij}} + \boldsymbol{\Lambda}(\boldsymbol{\lambda}_0, t_{ij})\boldsymbol{u}_i \end{aligned} \tag{5}$$

Exponential Growth Curve Model as a Special Case of Structured Latent Curve Model

(Browne, 1993; Browne & du Toit, 1991)

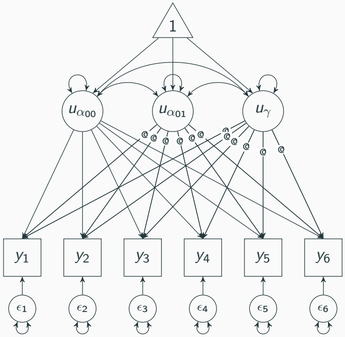

Figure 5: Exponential growth curve model
@ = loadings composed of constraint functions with estimated parameters

$$
\begin{aligned}
y_{ij} &= f(\boldsymbol{\lambda}_i, t_{ij}) + \epsilon_{ij}, \\
\boldsymbol{\lambda}_i &= \boldsymbol{\lambda}_0 + \boldsymbol{u}_i \\
f(\boldsymbol{\lambda}_i, t_{ij}) &\approx f(\boldsymbol{\lambda}_0, t_{ij}) = \mu_{y_{ij}} \\
&+ \frac{d}{d\boldsymbol{\lambda}_0} f(\boldsymbol{\lambda}_0, t_{ij})(\boldsymbol{\lambda}_i - \boldsymbol{\lambda}_0)
\end{aligned}
$$

$$
\begin{aligned}
y_{ij} &= \alpha_{00i} + \alpha_{01i}[1 - \exp(-\gamma_i t_{ij})] \\
&+ \epsilon_{ij} \\
f(\boldsymbol{\lambda}_i, t_{ij}) &\approx \mu_{y_{ij}} + \frac{\partial \mu_{y_{ij}}}{\partial \mu_{\alpha_{\mathbf{00}}}} u_{\alpha_{\mathbf{00}},i} \\
&+ \frac{\partial \mu_{y_{ij}}}{\partial \mu_{\alpha_{\mathbf{01}}}} u_{\alpha_{\mathbf{01}},i} + \frac{\partial \mu_{y_{ij}}}{\partial \mu_{\gamma}} u_{\gamma,i} \\
&= \mu_{y_{ij}} + \boldsymbol{\Lambda}(\boldsymbol{\lambda}_0, t_{ij}) \boldsymbol{u}_i \\
\mu_{ij} &= \mu_{\alpha_{\mathbf{00}}} + \mu_{\alpha_{\mathbf{01}}}[1 - \exp(-\mu_{\gamma} t_{ij})]
\end{aligned}
$$

Summary of Single-Phase Growth Curve Models

Table 1: Comparisons of Key Characteristics across Models. NA = not applicable; GC = growth curve; ID = individual differences; BC = between-class differences; I = intercept; S = slope

	Models			
Features Features	(a) Linear GC	(b) Latent basis	(c) Growth mixture	(d) Exponential GC
Change point	No	Not explicit	No	No
ID in change point	No	Not explicit	No	No
Possibility to stay within regimes	NA	NA	NA	NA
Sources of ID	ID in I & S	ID in I & S	ID and BC in I & S	ID in (non)linear growth parameters

Piecewise/Multi-Phase Growth Curve Models

Why Muiti-phase Models?

There are multiple reasons why multi-phase models may be needed:

- Phases of regularity in change phenomena over time
- Complex phenomena, with a hierarchy of subprocesses
- Stagewise processes
- Sudden shifts in observed behavior/state – sometimes with continuous changes in modeling parameters
- Study-based phases (e.g., baseline, intervention, post-intervention)
- etc., etc.

Relevant Methodological Approaches

- Mixed effects models with multiple phases/change points (Bryk & Raudenbush, 1992; Cudeck & Klebe, 2002; Diggle, Liang, & Zeger, 1994; Hall et al., 2000, 2001; McArdle, Ferrer–Caja, Hamagami & Woodcock, 2002)
- Mixture Structural Equation models with regime switching/latent transition (Dolan et al., 2005; Schmittmann et al., 2005; Muthén, 2001; Muthén & Asparouhov, 2011)
- Hidden Markov model (Elliott, Aggoun, & Moore, 1995)
- Latent transition analysis and latent profile analysis Clogg (1995); Collins and Wugalter (1992); Langeheine and Rost (1988); Lanza, Patrick, and Maggs (2010); Lazarsfeld and Henry (1968)
- Regime switching/Markov switching models (Frühwirth-Schnatter,2006; Hamilton, 1989, 1993; Kim & Nelson, 1999; Tong & Lim, 1980; Tsay, 1989)
- Bifurcation in nonlinear dynamical systems models (e.g., catastrophe models; Molenaar & van der Maas, 1992)

Multi-Phase Polynomial Growth Curve Model

(Cudeck, 2002; Harring et al., 2006; Smith, 1979)

- A multi-phase polynomial growth curve model may consist of $k = 1, \ldots K$ change points, denoted as $\tau_1, \ldots, \tau_K$, that partition each individual's repeated measurement data into $K + 1$ segments (p. 48, Cudeck, 2002; Smith, 1979).
- An example of a bilinear model – two-phase model with linear trajectory within each phase:

$$\begin{aligned} y_{ij} &= \alpha_{00i} + \alpha_{01i} t_{ij} + \alpha_{10i} \min(1, \max(0, t_{ij} - \tau_k)) + \alpha_{11i} \max(0, t_{ij} - \tau_k) \\ &= \begin{cases} \alpha_{00i} + \alpha_{01i} t_{ij}, & t_{ij} \leq \tau_1 \\ \alpha_{00i} + \alpha_{01i} t_{ij} + \alpha_{10i} + \alpha_{11i}(t_{ij} - \tau_1), & t_{ij} > \tau_1 \end{cases} \end{aligned} \tag{6}$$

- Harring, Cudeck, and du Toit (2006) provided an alternative expression for the max and min operators to enable the model be fitted as a SEM:

$$\max(l, v) = \frac{1}{2}\left(l + v + \sqrt{(l - v)^2}\right); \text{ and } \min(l, v) = \frac{1}{2}\left(l + v - \sqrt{(l - v)^2}\right)$$

Bilinear Model with Joint Level at Change Point

(Harring et al., 2006)

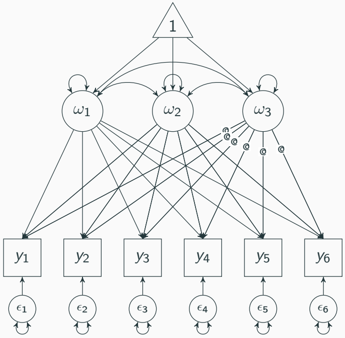

Figure 6: Bilinear joint level model @ = loadings composed of constraint functions with estimated parameters

- Harring et al. (2006) defined three alternative latent variables, denoted as ω_{i1}, ω_{i2}, and ω_{i3}, that are related to the parameters in Equation (6) as:

$$\alpha_{00i} = \omega_{i1} + \omega_{i3}\tau_1, \alpha_{01i} = \omega_{i2} - \omega_{i3},$$
$$\alpha_{11i} = \omega_{i2} + \omega_{i3}.$$

The overall function, for y_{ij} may then be expressed as a single function as:

$$y_{ij} = \omega_{i1} + \omega_{i2}t_{ij} + \omega_{i3}\sqrt{(t_{ij} - \tau_1)^2} \quad (7)$$

Latent Class Bilinear Model with Joint Level at Change Point

(Kohli et al., 2013)

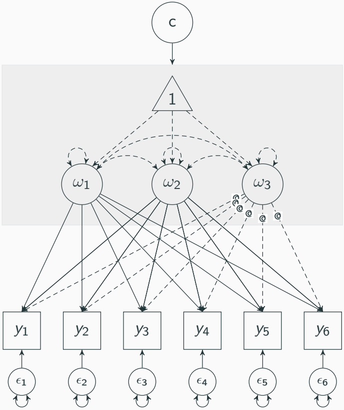

Figure 7: Latent class bilinear model

- Kohli, Harring, and Hancock (2013) extended the bilinear model with joint level to include a latent class variable.
- Allows for class-specific: (1) $E(\omega_{i1})$, $E(\omega_{i2})$ $E(\omega_{i3})$; and (2) τ_1
- $\tau_{1,C_i} \geq \max(T_i)$ allows individuals to never transition

Bilinear Model with Individual Differences in Change Point via the Structured Latent Curve Approach (Grimm, Ram, & Estabrook, 2016; Preacher & Hancock, 2015)

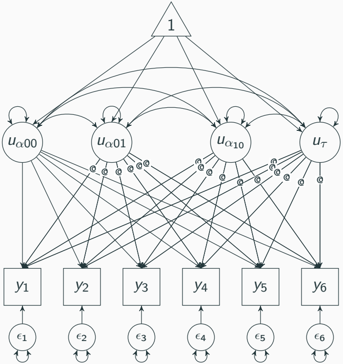

Figure 8: Bilinear Model with τ_{1i} via the Structured Latent Curve Approach

$$y_{ij} = \alpha_{00i} + \alpha_{01i} t_{ij} + \alpha_{11i} \max(0, t_{ij} - \tau_{1i})$$

The population curve is:

$$\begin{aligned} \mu_{y_{ij}} &= \mu_{\alpha_{00}} + \mu_{\alpha_{01}} t_{ij} \\ &+ \frac{\mu_{\alpha_{11}}}{2}\left(t_{ij} - \mu_{\tau_1} + h\right) \end{aligned}$$

where $h = \sqrt{(\mu_{\tau_1} - t_{ij})^2}$.
Individual growth curve obtained from applying the Taylor series expansion yields:

$$\begin{aligned} y_{ij} &= \mu_{y_{ij}} + \frac{\partial \mu_{y_{ij}}}{\partial \mu_{\alpha_{00}}} u_{\alpha_{00},i} + \frac{\partial \mu_{y_{ij}}}{\partial \mu_{\alpha_{01}}} u_{\alpha_{01},i} \\ &+ \frac{\partial \mu_{y_{ij}}}{\partial \mu_{\alpha_{11}}} u_{\alpha_{11},i} + \frac{\partial \mu_{y_{ij}}}{\partial \mu_{\tau}} u_{\tau_1,i} \qquad (9) \end{aligned}$$

Summary of Multi-Phase Growth Curve Models

Table 2: Comparisons of Key Characteristics across Models. NA = not applicable; GC = growth curve; ID = individual differences; BC = between-class differences; BR = between-regime differences; I = intercept; S = slope; LC = latent class;SPGM = Sequential process growth mixture

	Models				
Features	(e) Bilinear	(f) LC bilinear	(g) RS linear GC	(h) SPGM	(i) RS bilinear w/ τ_{1i}
Change point	Yes, $\hat{\mu}_\tau$	$\hat{\mu}_\tau$ by class	Ongoing RS	Fixed & known	$\hat{\mu}_\tau$; ongoing RS
ID in change point	No	Only BC	C_{ij}	No, but C_{i,τ_1}	BR; Var(τ_i); C_{ij}
Stay in phase 1 OK?	No	Maybe; fix $\tau_{class} > T$	Yes	Yes	Yes
Sources of ID	ID in I & Ss	ID & BC in I & Ss	ID & BR in Is & Ss	See (g)	ID & BR in I & Ss; ID in τ_{1i}

Regime Switching (RS) Extensions

What If the Latent Classes Are Also Dynamic Over Time?

- Individuals can switch between classes over time. These classes can be thought of as latent *regimes* or *phases*
- Each "regime" can be thought of as one of the stages or phases of a dynamic process.
- Unlike hidden Markov or latent transition models, a submodel is used to described the distinct change patterns associated with each phase.
- The changes that unfold within a regime can be continuous in nature.

Conceptual Sketch of Individuals Who Do Not Benefit from Intervention – What If Such Class Membership Is Dynamic?

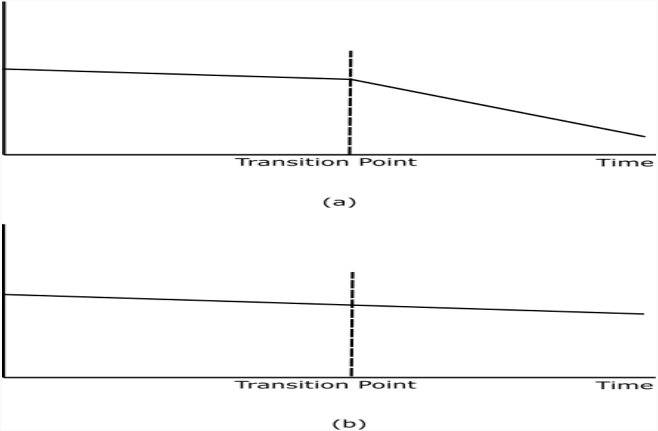

Figure 9: Figure illustrating (a) the trajectory of an individual who enters the second phase in a regime switching model compared to (b) one who does not enter the second phase.

Sequential Process Growth Mixture (SPGM) Model (Kim & Kim, 2012)

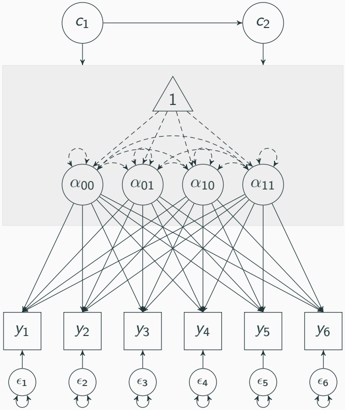

Figure 10: Sequential process growth mixture model

- Subjects are hypothesized to transition through two growth curve phases
- The transition (change) point is fixed and known
- Trajectories may be discontinuous in levels at the change point (i.e., a "leap" is possible)

$$y_{ij} = \begin{cases} \alpha_{00i} + \alpha_{01i} t_{ij}, \\ \alpha_{00i} + \alpha_{01i} t_{ij} + \alpha_{10i} + \alpha_{11i}(t_{ij} - \tau_1), \end{cases} \tag{10}$$

τ_1 assumed fixed and known

Regime-Switching (RS) Linear Growth Curve Model

(Dolan, Schmittmann, Lubke, & Neale, 2005; Schmittmann, Dolan, van der Maas, & Neale, 2005)

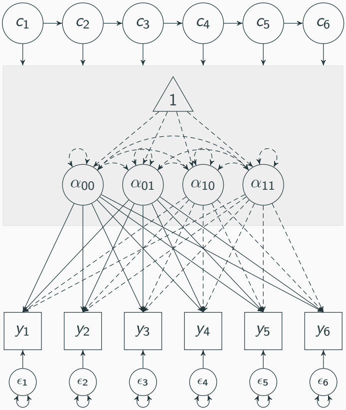

Figure 11: RS linear growth curve

- A latent regime indicator exists for each time point
- Ongoing transition through different regimes possible
- τ_1 could be estimated if needed
- One helpful special case is to define a regime in which the loadings of α_{10} and α_{11} on the ys, $E(\alpha_{10})$ and $E(\alpha_{11})$, and possibly their variances and covariances are all zero.

$$y_{ij} = \begin{cases} \alpha_{00i} + \alpha_{01i} t_{ij}, & C_{ij} = 1 \\ \alpha_{00i} + \alpha_{01i} t_{ij} + \alpha_{10i} & \\ +\alpha_{11i}(t_{ij} - \tau_1), & C_{ij} = 2 \end{cases} .$$

RS Bilinear Model with Individual Differences in Change Point via the Structured Latent Curve Approach (Grimm et al., 2016; Preacher & Hancock, 2015)

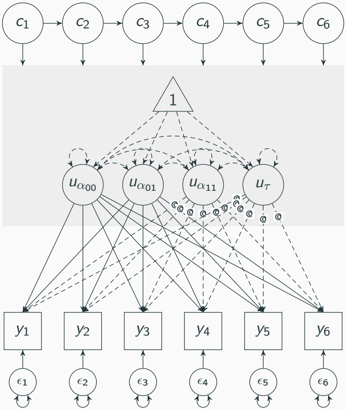

Figure 12: RS linear-linear model with τ_{1i}

$$
\begin{aligned}
y_{ij} &= \alpha_{00i} + \alpha_{01i} t_{ij} \\
&+ \alpha_{11i} \max(0, t_{ij} - \tau_{1i}) + \epsilon_{ij}.
\end{aligned}
$$

- Allows for the existence of bilinear trajectories, between-individual differences in change point, and switching between different "classes" of growth curve trajectories on an ongoing basis.
- Individuals could start out, for example, in a low-performing class but later switch to a high-performance class, or return to the first class.

Summary of Multi-Phase Growth Curve Models

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Stay in phase 1 OK?	No	Maybe; fix $\tau_{class} > T$	Yes	Yes	Yes
Sources of ID	ID in I & Ss	ID & BC in I & Ss	ID & BR in Is & Ss	See (g)	ID & BR in I & Ss; ID in τ_{1i}

Motivating Example: Data from the Early Childhood Longitudinal Study, Kindergarten Class of 1998-99 (U.S. Department of Education, National Center for Education Statistics, 2010)

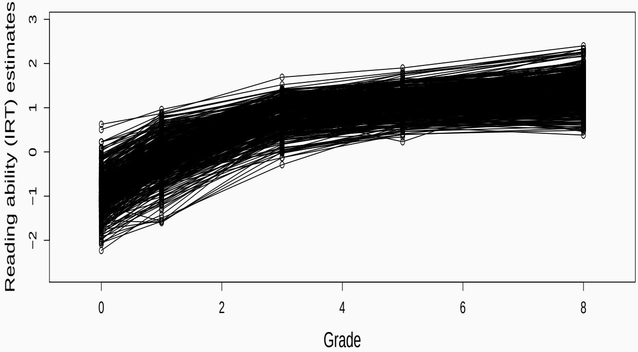


Figure 13: Plot of the item response theory (IRT)-scaled ability estimates of children's reading from the ECLSK study. A subsample of the data set ($n = 2369$) with longitudinal assessments of the children's reading skills from kindergarten through eighth grades was used.

Model Fitting Results

Table 4: Fit Measures from Models (a)—(i). GC = Growth Curve

	(a) Linear GC	(b) Latent Basis	(c) Growth Mixture	(d) Exponential GC	(e) Linear-linear joint level	(f) Latent Class Linear-Linear	(g) RS Linear GC	(h) Sequential Process Growth Mixture	(i) RS Linear-Linear w/ τ_{1i}
AIC	16469.62	526.11	16428.38	1082.06	309.41	-289.10	182.87	732.30	-325.05
BIC	16492.69	578.04	16468.77	1139.75	372.88	-179.47	269.41	824.61	-215.43
sBIC	16479.99	549.44	16446.53	1107.98	337.93	-239.84	221.75	773.78	-275.80
Entropy	–	–	0.93	–	–	0.77	0.97	0.84	0.88

- We imposed special constraints in the RS linear growth curve model (Model g) to capture a "high-risk" regime characterized by no difference in slope in phase 2, and individuals who had a high probability of staying within this regime.
- The final Model g was found to have the third lowest IC values among all models considered, and excellent entropy value (.97).
- The latent class linear-linear model (Model f) and the RS linear-linear model with τ_{1i} showed the best fit in terms of IC values of residual error variances, but less satisfactory entropy values.

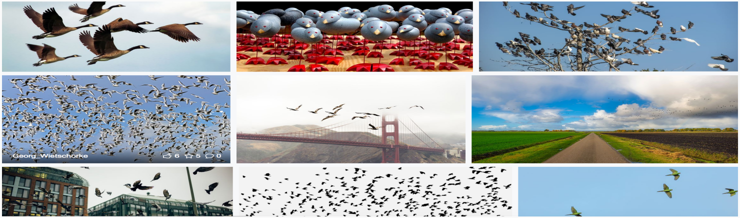
What Else Do the Model Estimates Tell Us?

- The estimated transition probability matrix from the RS linear growth curve model suggested that the low initial ability regime was highly unstable.
- The estimated posterior regime probabilities from this model indicated that only 7% of the subsample started out in the low initial ability regime
- The discrepancies in initial reading ability had equalized considerably by the end of first grade.
- The latent class linear-linear model (Model f) does not allow for transition in class membership. The two extracted classes ended up with poor separation.
- The RS linear-linear model (Model i) was overparameterized for these data – $Var(\tau_i)$ was close to 0.

What Else Do the Model Estimates Tell Us?

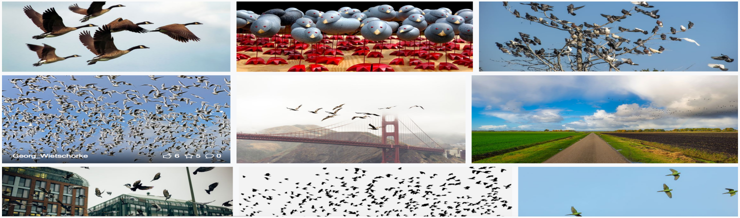
- Reliance on the estimated posterior latent class/regime probabilities to draw inference about class membership can be misleading as their accuracy depend considerably on the meanings of, and relative separation between the regimes.
    - Conclusions regarding when the low initial ability class diminishes differed between the RS linear growth curve model and the RS linear-linear model with τ_i even though the former can be obtained as a special case of the latter.
- Fixing the change point at a constant value (as in the sequential process mixture model) yielded bad fit.

Summary, Challenges, and Opportunities



- Regime-switching longitudinal models offer a way to study differences in change dynamics *within* as well as *between* phases of change.
- In the illustrative example, I used growth curve models to represent the change dynamics within each regime. Other structural equation models and dynamics models (e.g., state-space models, differential equation models) can also be used.
- The highly constrained nature of the model offers a confirmatory approach to investigate preconceived differences in dynamics.

Summary, Challenges, and Opportunities



- Challenges and unresolved issues:
  - Estimation difficulties: local minima/maxima and computational costs with large T
  - Other alternative regime-switching approaches may be used in scenarios with large T (e.g., *dynr;* Chow et al., 2018; Ou, Hunter, & Chow, 2018, revised and resubmitted)
  - Issues in classifying individuals and time points into regimes based on Pr(regime of person i at time t|data)

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